**3B**

**5.solution: assume that (e1,e2,e3,e4) is a basis of R4, we define T∈L(R4,R4) that: Te1=Te2=0,Te3=e1,Te4=e2.note that nullT=rangeT=span(e1,e2), hense the nullT=rangeT.**

**6.see the solution**

**10.conclusion: suppose (v1,...,vn) spans V and T∈L(V,W), prove that (Tv1,...,Tvn)spans rangeT.**

**Similar conclusion: suppose (v1,...,vn) spans V and T∈L(V,W) is surjective, prove that (Tv1,...,Tvn)spans W.**

**point: prove that rangeT⊂span(Tv1,...,Tvn) and rangeT⊃span(Tv1,...,Tvn)**

**12.solution: there exsists a subspace U of V that V = U⊕nullT. From the defination of direct sum we have U∩nullT={0}.below, we should proof that rangeT={Tu:u∈U}. obviously, {Tu:u∈U}⊂rangeT. To proof the other hand, suppose that v∈V. There exists u`∈U and w∈nullT such that v=u`+w. Applying T to both sides so that Tv=T(u`+w)=Tu`+Tw=Tu` which means Tv∈{Tu:u∈U}. cause Tv is an arbitrary vector in rangeT(since v is an arbitrary vector in V)， rangeT⊂{Tu:u∈U}.(here in my own opinion,we should proof that Tv can be linear combined by a vector in {Tu:u∈U} so that Tv is a vector in {Tu:u∈U}, and Tv must be a arbitrary vector in rangeT) hence rangeT={Tu:u∈U}.**

**13.Solution: Since T∈L(F4,F2), we know that ((1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)) is a basis of V. since nullT is a subspace of v, (x1,x2,x3,x4) can be rewritten below: (x1,x2,x3,x4)=x2(5,1,0,0)+x4(0,0,7,1). Hence ((5,1,0,0),(0,0,7,1)) is a basis of nullT in the condition of x2 and x4 are arbitrary. Thus dimnullT=2 and dimrangeT=dimV-dimnullT=2 which means rangeT=F2. Thus rangeT=W and T is surjective as desire.**

**16.Suppose that there exists a linear map on V whose null space and range are both finite dimensional. Prove that V is finite dimensional.**

**(Note that the hypothesis that V is finite dimensional which we are trying to prove in this exercise can not be used here.)**

**key point: we must find a list (v1,…,vn) in V such that span(v1,…,vn)=V(by the definition of finite dimensional vector space)**

**Proof: Suppose T is a linear map from V to W. Hence there exist a list of vectors u1,…,un∈nullT and w1,…,wm∈rangeT that (u1,…,un) spans nullT and (w1,…,wm) spans rangeT. Since each wj(1≤j≤m)∈nullT, there exist vectors v1,…,vm ∈V that Tvj=wj. Suppose v∈V, then Tv∈rangeT. So there exist a1,…,am∈Fsuch that Tv=∑1≤j≤majwj=∑1≤j≤majTvj=T(∑1≤j≤majvj). We can rewrite it in the way that T(v-∑1≤j≤majvj)=0 which means v-∑1≤j≤majvj∈nullT. Thus there exist b1,…,bn∈F such that v-∑1≤j≤majvj=∑1≤k≤nbkuk. Hence we have v=∑1≤j≤majvj+∑1≤k≤nbkuk, in other words, (a1,…,am,b1,…,bn) spans V. Hence V is finite dimensional.**

**17. Suppose V and W are both finite dimensional. Prove that there exists an injective linear map from V to W if and only if dimV≤dimW.**

**Proof: Prove the one hand first that if there exists an injective linear map from V to W, dimV≤dimW. Suppose T∈L(V,W). Because V and W are both finite dimensional and T is injective which means nullT={0}, dimV=dimnullT+dimrangeT=dimrangeT≤dimW(since rangeT is a subspace in W). Let`s prove the other hand. ~~Suppose dimV=n and dimW=m, we have n≤m. Suppose w~~~~1~~~~,…,w~~~~n~~ ~~is a linearly independent list of W. Define T by: T(z~~~~1~~~~,…,z~~~~n~~~~)= z~~~~1~~~~w~~~~1~~~~,+…+w~~~~n~~~~z~~~~n~~~~(z~~~~1~~~~,…,z~~~~n~~~~∈F). By 3(b), because w~~~~1~~~~,…,w~~~~n~~ ~~is linearly independent, T(0,…,0)=0 if and only if z~~~~1~~~~=...=z~~~~n~~~~, which means nullT={0}. Hence, T is injective.~~**

**Suppose (v1,…,vn) is a basis of V and (w1,…,wm) is a basis of W. Define T by: Tvj=wj, 1≤j≤n. Because n≤m, Tvj=wj (1≤j≤n) makes sense. Suppose v∈nullT. Because (v1,…,vn) is a basis of V, there exist a1,…,an∈F such that v=∑1≤j≤najvj. Applying T for both sides: Tv=T(∑1≤j≤najvj)=∑1≤j≤najTvj=∑1≤j≤najwj. Cause w1,…,wm is linearly independent which means w1,…,wn is linearly independent either, 0=Tv=∑1≤j≤najwj if and only if aj=0. Thus nullT={0}, in other words, T is injective as desire.(Here if we define T as T(z1,…,zn)=∑1≤j≤nzjwj, then it will be specialized since T is a linear map from Fn to W.)**

**18.Suppose V and W are both finite dimensional. Prove that there exists a surjective linear map from V to W if and only if dimV≥dimW.**

**23.Suppose V and U are finite-dimensional vector spaces and S∈L(V,W) and T∈L(U,V). Prove that dimrangeST≤min{dimrangeT,dimrangeS}**